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ME 335 FLUID MECHANICS II

Department of Mechanical Engineering
University of Saskatchewan

Midterm Exam

Thursday, February 13, 2014, 1:00 – 3:00 p.m.

Instructor: Prof. David Sumner

PLEASE READ CAREFULLY:

This exam has 10 pages. The exam is "closed book." An equation sheet is provided for your use during the exam. No calculators are allowed. No cell phones or any other electronic devices are permitted.

There are 4 problems on the exam worth a total of 100 marks. Attempt all 4 problems. The number of marks assigned to each part of a problem is indicated in the left hand column. Answers are to be written on the exam question sheets in the spaces provided. If you need more room, use the back of the sheet. Please ensure that your answers are clear and legible. Please put a box around, or underline, your final answers (where appropriate).

Name: SOLUTIONS

Student No.: __________

Signature: __________

Problem	Relevant Learning Outcomes	Grade
1	LO1, LO2, LO3, LO4	/15
2	LO2, LO3, LO4	/15
3	LO1, LO2	/30
4	LO3	/40
	Total:	/100

Relevant Learning Outcomes:

LO1: Use the differential equations of fluid motion in Cartesian and cylindrical coordinates to calculate the fluid acceleration, velocity, vorticity, pressure, and shear stress fields.

LO2: Classify a flow as inviscid, incompressible, irrotational, steady, and fully developed.

LO3: Simplify the differential equations for the conservation of mass, momentum, and energy for steady incompressible Newtonian viscous flow, and prescribe appropriate boundary conditions, to solve for the velocity profile, pressure gradient, and mass flow rate in problems involving Couette flow, Poiseuille flow, channel flow, and pipe flow.

LO4: Describe the fundamental properties of laminar and turbulent flows and explain the transition process from laminar to turbulent flow.

Problem 1 (15 marks)

Fill in the blanks below.

— $\nabla \cdot \vec{V} = 0$

1 a. An "incompressible flow" is characterized by —constant density ________.

—negligible changes in density
—negligible compressibility effects

1 b. A flow may be considered "inviscid" when — effects of viscosity are negligible

— viscous shear stress contributions are small or negligible or vanish
— viscous terms in N-S equations cancel out
* $\mu = 0$ is physically impossible and does not define an inviscid flow.

– curl of the velocity

2 c. The – vorticity vector represents the local fluid rotation; if its components are all zero, then the fluid is said to be irrotational.

1 d. "Fully developed flow" in a pipe is a flow in which – the axial velocity profile no longer changes along the pipe

– the flow is no longer influenced by entrance effects

1 e. In the differential equation for conservation of energy, the term that represents the "convection of heat by the local velocity field" is given by $\rho c(\vec{V} \cdot \nabla) T$

2 f. any 2 from list at right and ________ are two different physical characteristics of all turbulent flows.

- fluid elements starting at the same place but at different times follow different paths
- rapid, random fluctuations in time and space
- enhanced levels of mixing
- unstable – amplification of small, occasional naturally occurring disturbances
- continuous fluctuations that encompass a range of frequencies

2 g. The dimensionless Reynolds number represents the ratio between inertial forces and viscous forces.

1 h. After applying the "Reynolds decomposition" and then time-averaging the Navier-Stokes equations, some additional terms, known as the –Reynolds stresses, appear in the equations for turbulent flows.

–Reynolds shear stresses
–turbulent stresses

2 i. The solution for <u>shear-driven</u> viscous flow between two parallel plates is known as Couette flow, whereas the solution for <u>pressure-driven</u> flow between two parallel plates is known as Poiseuille flow.

2 j. In steady, laminar viscous, fully developed, incompressible, Newtonian fluid flow (with constant properties) along a circular pipe, the pressure drop (ΔP) is directly proportional to μ, U_{avg}, L, Q, τ_{WALL} but inversely proportional to R, D, A.

ρ for non-horiz. pipe
g for non-horiz. pipe

Problem 2 (15 marks)

Provide short, descriptive answers to the following questions.

5 a. For the fluid flow between two long concentric rotating cylinders, briefly explain what happens when the critical value of the Taylor number, Ta, is reached, i.e., when

$$\mathrm{Ta} = \frac{r_i (r_o - r_i)^3 \Omega_i^2}{\upsilon^2} \approx 1700,$$

where r_i is the radius of the inner cylinder, r_o is the inside radius of the outer cylinder, Ω_i is the angular velocity of inner cylinder, and v is the kinematic viscosity.

- See pp. 275–276 of the textbook – assigned reading
- see Figure 4.14 of the textbook
- at the critical value of the Taylor number, the rotating flow becomes unstable – flow instability arises
  - plane flow disappears
  - three-dimensional laminar flow pattern arises
  - rows of alternating toroidal "Taylor vortices"
  - instability seen in the exact solution to the Navier–Stokes equation for this flow
  - not related to transition from laminar to turbulent flow – vortices are still laminar

5 b. What is the "Reynolds decomposition," as applied to the statistical analysis of a turbulent velocity time-signal, $u(t)$, over a time period, T? How is the turbulence intensity, TI_u, defined?

- velocity-time signal split up into two components or contributions:

$$u(t) = \bar{U} + u'(t)$$ 2

mean or time-averaged component

fluctuations component (with reference to the mean)

- where $\bar{U} = \frac{1}{T}\int_0^T u(t)\,dt$ for averaging period T 1

- turbulence intensity expressed as the root-mean-square of the velocity fluctuations relative to the mean: 1

$$\mathrm{TI}_u = \frac{1}{\bar{U}}\left[\frac{1}{T}\int_0^T \left[u'(t)^2\right]dt\right]^{1/2}$$ 1

Problem 2 continues on the following page

Problem 2 (continued)

5 c. Dimensionless pressure gradient data for fully developed flow in a circular pipe are shown in the figure below. What is the significance of the discontinuity in the graph? For the same pipe, which fluid would be moving faster at this point of discontinuity, air or water (at "standard conditions")?

- the discontinuity marks the transition from laminar to turbulent flow, and the upper limit of validity for the exact solution of the Navier-Stokes equation for Hagen-Poiseuille flow 4
- the discontinuity occurs at a critical value of the Reynolds number

$$Re_c = \left(\frac{\rho U_{avg} \cancel{D}}{\mu}\right)_{H_2O} = \left(\frac{\rho U_{avg} \cancel{D}}{\mu}\right)_{AIR}$$

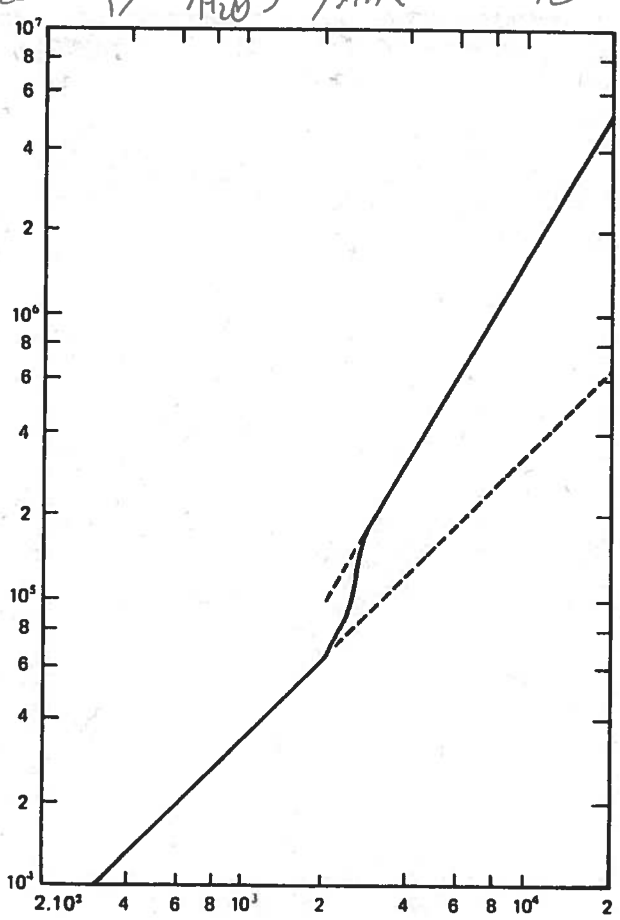
- approximate values of ρ and μ for H_2O and air

$$\frac{(U_{avg})_{AIR}}{(U_{avg})_{H_2O}} = \left(\frac{\rho}{\mu}\right)_{H_2O}\left(\frac{\mu}{\rho}\right)_{AIR} \sim \frac{10^3}{10^{-3}} \cdot \frac{10^{-5}}{10^0} \sim \frac{10^{-2}}{10^{-3}} \sim 10$$

- $\boxed{\text{air}}$ would be moving faster by a factor of ~10

1

$$\frac{dP}{dx}\frac{\rho D^3}{\mu^2}$$



$$Re = \frac{\rho u_{avg} D}{\mu}$$

Re = 2000, D = 1m

H_2O

$$U = \frac{2000 \cdot 1\times10^{-3}}{1000 \cdot 1} = 0{,}002 \text{ m/s}$$

AIR

$$U = \frac{2000 \cdot 1.8\times10^{-5}}{1.1} = 0{,}036 \text{ m/s}$$

Source: Tritton, D.J., 1988, Physical Fluid Dynamics, 2nd Edition, Oxford: Oxford University Press, page 20.

Problem 3 (30 marks)

The following velocity field represents an unsteady, compressible, two-dimensional flow of a Newtonian fluid. The fluid density is <u>only a function of time</u>, i.e., $\rho = \rho(t)$ only.

$$u = \frac{x}{1+t} \qquad v = \frac{2y}{2+t} \qquad w = 0$$

15

a. Find an expression for the fluid density, $\rho(t)$. – solve using continuity – similar to assignment problems

– for unsteady, compressible flow, the continuity equation is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$ 1

– here, $\rho = \rho(t)$ only, so this equation becomes:

$$\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{v}) = 0$$

$$\frac{d\rho}{dt} + \rho\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}}^{\,0,\ w=0}\right] = 0$$ 2

$$\frac{d\rho}{dt} + \rho\left[\frac{\partial}{\partial x}\left(\frac{x}{1+t}\right) + \frac{\partial}{\partial y}\left(\frac{2y}{2+t}\right)\right] = 0$$

$$\frac{d\rho}{dt} + \rho\left[\frac{1}{1+t} + \frac{2}{2+t}\right] = 0$$ 2

$$\frac{d\rho}{dt} = -\rho\left[\frac{1}{1+t} + \frac{2}{2+t}\right]$$ – separable ODE (solve by integration)

$$\int \frac{d\rho}{\rho} = \int -\left[\frac{1}{1+t} + \frac{2}{2+t}\right] dt$$ – similar integrals in lectures and on assignments

$$\ln(\rho) = -\ln(1+t) - 2\ln(2+t) + C_1$$ 5

$$\ln(\rho) = \ln\left(\frac{1}{1+t}\right) + \ln\left(\frac{1}{(2+t)^2}\right) + \ln C_2$$

or $$\boxed{\rho(t) = \frac{C_2}{(1+t)(2+t)^2}}$$ 5

must solve for $\rho(t)$ explicitly

Problem 3 continues on the following page

Problem 3 (continued)

5 b. Find the local acceleration of the fluid.

$$\vec{a} = \frac{d\vec{V}}{dt} = \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{LOCAL ACCELERATION}} + \underbrace{(\vec{V}\cdot\nabla)\vec{V}}_{\text{CONVECTIVE ACCELERATION}} \qquad 1$$

$$\frac{\partial \vec{V}}{\partial t} = \frac{\partial u}{\partial t}\hat{\imath} + \frac{\partial v}{\partial t}\hat{\jmath}$$

$$= \frac{\partial}{\partial t}\left(\frac{x}{1+t}\right)\hat{\imath} + \frac{\partial}{\partial t}\left(\frac{2y}{2+t}\right)\hat{\jmath}, \text{ giving...}$$

$$\boxed{\vec{a}_{LOCAL} = \underbrace{\frac{-x}{(1+t)^2}\hat{\imath}}_{2} - \underbrace{\frac{2y}{(2+t)^2}\hat{\jmath}}_{2}.} \qquad 1$$

-acceleration is a <u>vector</u>

10 c. Find the convective acceleration of the fluid.

$$\vec{a}_{CONVECTIVE} = (\vec{V}\cdot\nabla)\vec{V} \qquad 1$$

$$= (\vec{V}\cdot\nabla)u\,\hat{\imath} + (\vec{V}\cdot\nabla)v\,\hat{\jmath}$$

$$(\vec{V}\cdot\nabla)u = u\frac{\partial u}{\partial x}^{1} + v\frac{\partial u}{\partial y}^{1}$$

-note that $(\vec{V}\cdot\nabla) \neq \nabla\cdot\vec{V}$.

$$= \left(\frac{x}{1+t}\right)\frac{\partial}{\partial x}\left(\frac{x}{1+t}\right) + \left(\frac{2y}{2+t}\right)\frac{\partial}{\partial y}\left(\frac{x}{1+t}\right)$$

$$= \left(\frac{x}{1+t}\right)\left(\frac{1}{1+t}\right) + 0 = \frac{x}{(1+t)^2} \qquad 2$$

$$(\vec{V}\cdot\nabla)v = u\frac{\partial v}{\partial x}^{1} + v\frac{\partial v}{\partial y}^{1}$$

$$= \left(\frac{x}{1+t}\right)\frac{\partial}{\partial x}\left(\frac{2y}{2+t}\right) + \left(\frac{2y}{2+t}\right)\frac{\partial}{\partial y}\left(\frac{2y}{2+t}\right)$$

$$= 0 + \left(\frac{2y}{2+t}\right)\left(\frac{2}{2+t}\right) = \frac{4y}{(2+t)^2} \qquad 2$$

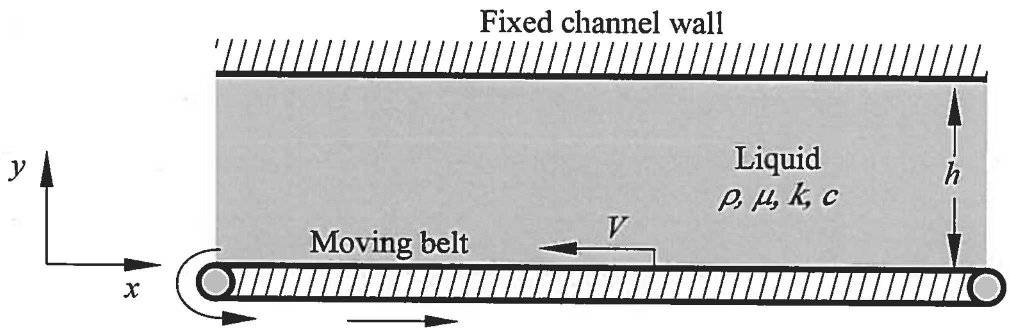
$$\therefore \boxed{\vec{a}_{CONV} = \frac{x}{(1+t)^2}\hat{\imath} + \frac{4y}{(2+t)^2}\hat{\jmath}}$$

-acceleration is a vector

1

Problem 4 (40 marks)

Pressure is driving a viscous liquid from left to right along the narrow horizontal channel shown in the figure below. The upper wall of the channel is fixed. The pressure-driven flow is counteracted by a moving belt that forms the lower wall of the channel, as shown in the figure. The belt moves from right to left at a constant velocity, V. Gravity may be neglected.



The flow is steady, viscous, incompressible, and laminar. The height of the channel is h. The liquid is Newtonian with constant properties, ρ, μ, k, and c. The flow is two-dimensional and purely axial, i.e. $v = w = 0$.

5 a. Show that if the flow is purely axial, then it follows that the flow must also be fully developed.

- if flow is two-dimensional, then $w=0$ and no z-variation
- if flow is purely axial, then $v=w=0$
- continuity equation for incompressible flow:

$$\nabla \cdot \vec{V} = 0 \qquad 1$$

$$\frac{\partial u}{\partial x} + \underbrace{\cancel{\frac{\partial v}{\partial y}}}_{0,\ v=0} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0,\ w=0} = 0 \qquad 1$$

PURELY AXIAL FLOW 1   PURELY AXIAL FLOW 1

$\therefore \frac{\partial u}{\partial x} = 0$ → velocity profile does not change along the pipe and $u(y)$ only → fully developed flow

1

- shown using continuity equation
- note that steady flow ≠ fully developed flow

5 b. Specify the boundary conditions for the axial velocity profile, $u(y)$.

- no-slip boundary conditions:

3 $\boxed{u(0) = -V}$ MOVING BELT

2 $\boxed{u(h) = 0}$ FIXED UPPER WALL

- no other boundary conditions needed
- maximum velocity location will depend on V

Problem 4 continues on the following page

Problem 4 (continued)

10 c. Using the Navier-Stokes equations, show that the pressure is a function of x only, i.e. $P = P(x)$ only.

- Navier-Stokes equations:

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho(\vec{V}\cdot\nabla)\vec{V} = -\nabla P + \rho\vec{g} + \mu\nabla^2\vec{V}$$

- from the y-momentum equation:

2 $$\rho\cancel{\frac{\partial v}{\partial t}} + \rho\left(\cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}}\right) = -\frac{\partial P}{\partial y} + \cancel{\rho g_y} + \cancel{\mu}\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}}\right)$$

0, STEADY FLOW 1

0, $v=0$ PURELY AXIAL FLOW 1

0, $v=0$ PURELY AXIAL FLOW 1

0, NEGLECT GRAVITY 1

0, $v=0$ PURELY AXIAL FLOW 1

0, $v=0$ PURELY AXIAL FLOW 1

$$\therefore \frac{\partial P}{\partial y} = 0 \longrightarrow \underline{\underline{P(x) \text{ only}}}$$

2

- must make a specific statement or conclusion regarding the pressure

[similarly for z-momentum $\rightarrow \frac{\partial P}{\partial z} = 0$ (but flow is 2D)]

Problem 4 continues on the following page

Problem 4 (continued)

10 d. Using the Navier-Stokes equations, find an expression for the velocity profile, $u(y)$.

- from x-momentum equation:

$$\underbrace{\rho\cancel{\frac{\partial u}{\partial t}}}_{\text{0, STEADY FLOW}} + \rho\Big(\underbrace{\cancel{u\frac{\partial u}{\partial x}}}_{\text{0, FULLY DEVELOPED } 1} + \underbrace{\cancel{v\frac{\partial u}{\partial y}}}_{\text{0, } v=0 \text{ PURELY AXIAL } 1}\Big) = -\frac{\partial P}{\partial x} + \underbrace{\cancel{\rho g_x}}_{\text{0, NEGLECT GRAVITY}} + \mu\Big(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{\text{0, FULLY DEVELOPED } 1} + \frac{\partial^2 u}{\partial y^2}\Big)$$ 1

$$\therefore \frac{\partial P}{\partial x} = \mu\frac{\partial^2 u}{\partial y^2}$$ 2

- from (a), have $u(y)$ only
- from (c), have $P(x)$ only

$$\therefore \underbrace{\frac{dP}{dx}}_{\text{FUNCTION OF } x \text{ ONLY}} = \underbrace{\mu\frac{d^2u}{dy^2}}_{\text{FUNCTION OF } y \text{ ONLY}} = \text{constant}$$

→ "separation of variables"
→ only ordinary derivatives used 1

- integrate twice:

$$u(y) = \frac{1}{2\mu}\frac{dP}{dx}y^2 + C_1 y + C_2$$ 2

- apply boundary condition $u(0) = -V = C_2$
$\rightarrow C_2 = -V$ 1

- gives $u(y) = \frac{1}{2\mu}\frac{dP}{dx}y^2 + C_1 y - V$

- apply boundary condition:

$$u(h) = 0 = \frac{1}{2\mu}\frac{dP}{dx}h^2 + C_1 h - V$$

$$\rightarrow C_1 = \left(V - \frac{1}{2\mu}\frac{dP}{dx}h^2\right)\frac{1}{h} = \left(\frac{V}{h} - \frac{h}{2\mu}\frac{dP}{dx}\right)$$ 2

$$\therefore u(y) = \frac{1}{2\mu}\frac{dP}{dx}y^2 + \left(V - \frac{1}{2\mu}\frac{dP}{dx}h^2\right)\frac{y}{h} - V$$ 1

or

$$\boxed{u(y) = \frac{1}{2\mu}\frac{dP}{dx}\left(y^2 - yh\right) + V\left(\frac{y}{h} - 1\right)}$$

*recall that if improper/incorrect calculus appears ..... $d^2u = \frac{dP}{dx}\frac{1}{\mu}dy^2$ then, as mentioned in class, the entire question is zero

Problem 4 (continued)

10 e. Consider the special case where there is no pressure gradient, and the flow is solely driven by the belt. If the temperature profile is steady and fully developed, i.e. $T = T(y)$ only, and both walls are at the same temperature, T_{wall}, use the differential equation for the conservation of energy to find the temperature profile, $T(y)$.

– if there is no pressure gradient, then $\frac{dP}{dx} = 0$

– velocity profile simplifies to $u(y) = V\left(\frac{y}{h} - 1\right)$ 1

– viscous dissipation function, Φ, becomes:

$$\Phi = \mu\left(\frac{du}{dy}\right)^2 = \mu\left[\frac{V}{h}\right]^2$$ 2

– conservation of energy: $\rho c \frac{\partial T}{\partial t} + \rho c(\vec{V}\cdot\nabla)T = k\nabla^2 T + \Phi$ 1

$$\underbrace{\cancel{\rho c \frac{\partial T}{\partial t}}}_{\text{0, STEADY FLOW}} + \underbrace{\cancel{\rho c u \frac{\partial T}{\partial x}}}_{\text{0, FULLY DEVELOPED FLOW}} + \underbrace{\cancel{\rho c v \frac{\partial T}{\partial y}}}_{\text{0, V=0 PURELY AXIAL FLOW}} = \underbrace{\cancel{k\frac{\partial^2 T}{\partial x^2}}}_{\text{0, FULLY DEVELOPED FLOW}} + k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{V}{h}\right)^2$$

$$\therefore \frac{d^2T}{dy^2} = -\frac{\mu V^2}{kh^2}$$ 2

$$\frac{dT}{dy} = -\frac{\mu V^2}{kh^2}y + C_1$$ from integration

$$T(y) = -\frac{\mu V^2}{kh^2}\frac{y^2}{2} + C_1 y + C_2$$ from integration 2

– at $y=0$, $T = T_{WALL}$ → $C_2 = T_{WALL}$ 1

– at $y=h$, $T = T_{WALL}$ → $C_1 = +\frac{\mu V^2}{2kh}$ 1

$$\therefore \boxed{T(y) = \frac{\mu V^2}{2kh^2}\left(-y^2 + yh\right) + T_{WALL}}$$

* note that $\left(\frac{du}{dy}\right)^2 \neq \frac{d^2u}{dy^2}$!!