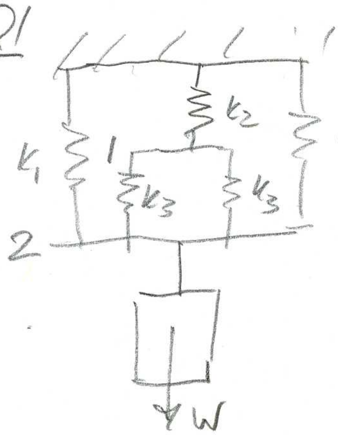
Q1



$$\Pi = \frac{1}{2}\left[2k_1(q_2-0)^2 + 2k_3(q_2-q_1)^2 + k_2(q_1-0)^2\right] - W q_2 \rightarrow \min$$

$$\frac{\partial \Pi}{\partial q_1} = \frac{1}{2}\left[4k_3(q_2-q_1)(-1) + 2k_2 q_1\right] = 0.$$

2

$$\frac{\partial \Pi}{\partial q_2} = \frac{1}{2}\left[4k_1 q_2 + 4k_3(q_2-q_1)\right] - W = 0$$

or
$$(2k_3+k_2)q_1 - 2k_3 q_2 = 0$$
$$-2k_3 q_1 + (2k_1+2k_3)q_2 = W$$
$$\Rightarrow \begin{bmatrix} k_2+2k_3 & -2k_3 \\ -2k_3 & 2k_1+2k_3 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ W \end{bmatrix}$$

a) if $k_1=k_2=k_3=k=30\,\frac{N}{mm}$ then

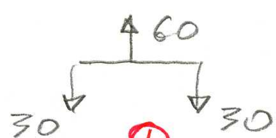
$$\left.\begin{aligned} k(3q_1-2q_2) &= 0. \\ k(-2q_1+4q_2) &= W \end{aligned}\right] \Rightarrow \begin{aligned} q_1 &= \frac{2}{8}\frac{W}{k} = 2\,mm \\ q_2 &= \frac{3}{8}\frac{W}{k} = 3\,mm \end{aligned}$$

2

Springs:
$$S_1 = k q_2 = \frac{3}{8}W = 90\,N$$
$$S_2 = k q_1 = \frac{2}{8}W = 60.$$
$$S_3 = k(q_2-q_1) = \frac{1}{8}W = 30$$

1

node 1



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node 2

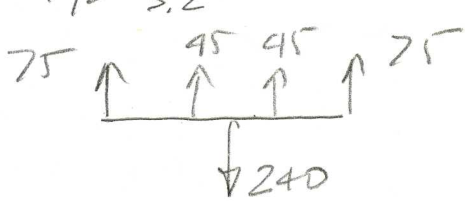
1

b) if $k_1=k_3=k$ and $k_2=3k$ then

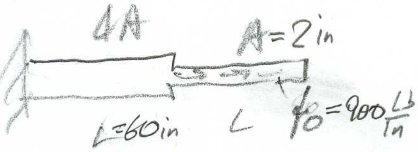
2

$$\left.\begin{aligned} k(5q_1-2q_2) &= 0. \\ k(-2q_1+4q_2) &= W \end{aligned}\right] \qquad \begin{aligned} q_1 &= \frac{W}{8k} = 1\,mm \\ q_2 &= \frac{W}{3.2k} = 2.5\,mm \end{aligned}$$

1 $\quad S_1 = k_1 q_2 = \frac{W}{3.2} = 75\,N$



Q2



$\tilde{u} = \alpha_0 + \alpha_1 x \qquad \tilde{u}(0) = \alpha_0 = 0 \qquad \tilde{u} = a_1\phi_1$ (1DOF)

$\Pi = \frac{1}{2}\int_0^{2L} AE\tilde{\varepsilon}^2 dx - \int_L^{2L} f_0 \tilde{u}\, dx \to \min$ 1 $\quad \phi_1 = x \quad \frac{d\phi_1}{dx} = 1$

$\tilde{\varepsilon} = \frac{d\tilde{u}}{dx} = a_1 \cdot 1$

a)

E=30·10^6

$\frac{\partial \Pi}{\partial a_1} = \int_0^L 4AE \underset{a_1}{\tilde{\varepsilon}} \underset{1}{\frac{\partial \tilde{\varepsilon}}{\partial a_1}} dx + \int_L^{2L} AE \underset{a_1}{\tilde{\varepsilon}} \underset{1}{\frac{\partial \tilde{\varepsilon}}{\partial a_1}} dx - \int_L^{2L} f_0 \underset{x}{\frac{\partial \tilde{u}}{\partial a_1}} dx = 0$ 2

$4AE \cdot a_1 x\Big|_0^L + AE\, a_1 x\Big|_L^{2L} - f_0 \frac{x^2}{2}\Big|_L^{2L} = 0$

$\frac{f_0 L^2}{EA} = \frac{900 \cdot 60^2}{30 \cdot 10^6 \cdot 2} = 0.054\,in$

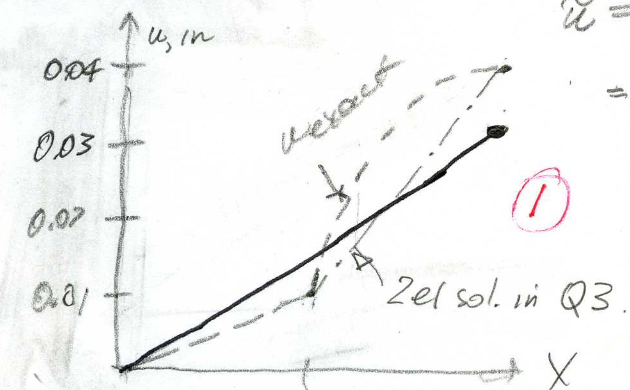
$\underbrace{5AEL}_{18\cdot 10^9}\, a_1 - \underbrace{f_0 \frac{3}{2}L^2}_{486\cdot 10^6} = 0 \qquad a_1 = \frac{3}{10}\overbrace{\frac{f_0 L}{AE}}^{0.0009} = 0.00027$

$\tilde{u} = \frac{3}{10}\frac{f_0 L}{AE} x$ 1 $\quad u(2L) = \frac{6}{10}\frac{f_0 L^2}{EA} = 0.0324\,in$ 1

$= 0.00027x$

$\tilde{\sigma} = E\tilde{\varepsilon} = E \cdot a_1 = \cancel{E} \cdot \frac{3}{10}\frac{f_0 L}{A\cancel{E}} = \frac{3}{10}\frac{f_0 L}{A} = \frac{3}{10}\frac{900 \cdot 60}{2}$

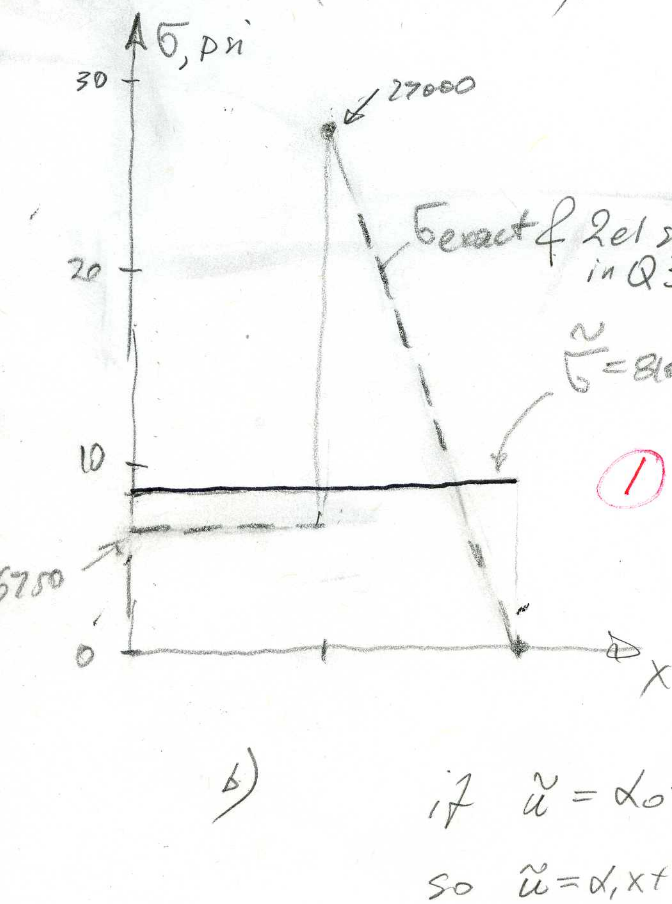
$= 8100\,psi$ 1



Exact:

$u_L = \frac{\hat{f_0 L} \cdot L}{4AE} + \frac{1}{2}\frac{\hat{f_0 L} L}{AE} = \frac{3}{4}\frac{f_0 L^2}{AE} = 0.0405$

$0.0135 + 0.027$



Accuracy:

Exact displacements are 'best' approximated by a line.

1 Exact stresses are 'best' approximated by a const.

Also: compare with 2 el. solution in Q3, if possible

b)

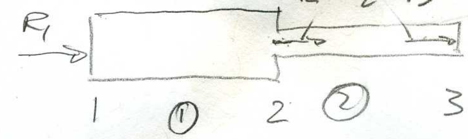
if $\tilde{u} = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 \qquad \tilde{u}(0) = \alpha_0 = 0$

so $\tilde{u} = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 = a_1 \underset{x}{\phi_1} + a_2 \underset{x^2}{\phi_2} + a_3 \underset{x^3}{\phi_3}$ 1

3DOF problem

Q3

$f_2^{(2)} = \frac{f_0 L}{2} = f_3^{(2)} = 27{,}000$



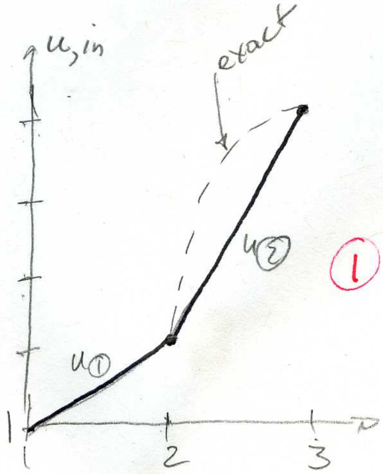
$K_{①} = \frac{4AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ $\quad K_{②} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Assembly:

$$\frac{AE}{L}\begin{bmatrix} 4 & -4 & \\ -4 & 4+1 & -1 \\ & -1 & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ \frac{f_0 L}{2} \\ \frac{f_0 L}{2} \end{bmatrix} \begin{matrix} \to R_1 \\ \to \text{solve} \end{matrix}$$

$\frac{AE}{L} \leftarrow 10^6$ $\qquad \frac{f_0 L}{2} \leftarrow 27{,}000$

2



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BC : $q_1 = 0$.

Solve

$$\frac{AE}{L}\begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} q_2 \\ q_3 \end{bmatrix} = \frac{f_0 L}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left.\begin{aligned} 5q_2 - q_3 &= \frac{f_0 L^2}{2EA} \\ -q_2 + q_3 &= \frac{f_0 L^2}{2EA} \end{aligned}\right] \quad 4q_2 = \frac{f_0 L^2}{EA}$$

1 $\qquad$ 2

$q_2 = \frac{f_0 L^2}{4AE} = 0.0135\,\text{in}$

$q_3 = \left(\frac{1}{2} + \frac{1}{4}\right)\frac{f_0 L^2}{AE} = \frac{3}{4}\frac{f_0 L^2}{AE} = 0.0405\,\text{in}$

$R_1 = -\frac{4AE}{L}\frac{f_0 L^2}{4AE} = -f_0 L = -54000$ ✓

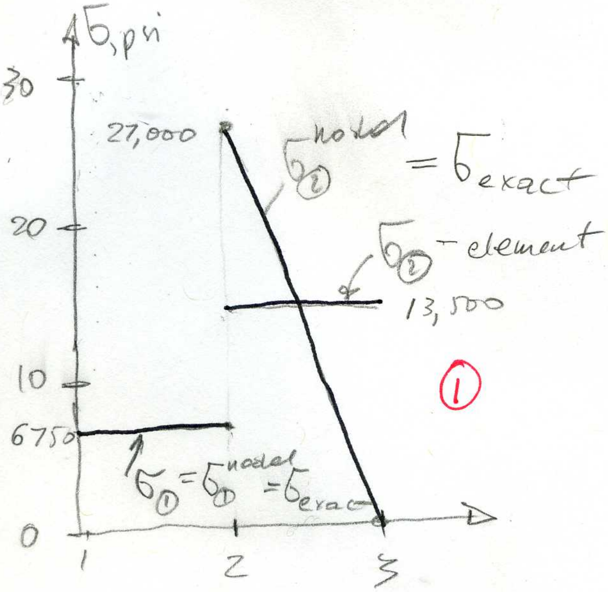
$\frac{f_0 L^2}{EA} = 0.054$

$\frac{f_0 L}{A} = \frac{900 \cdot 60}{2} = 27{,}000\,\text{psi}$

stress

el 1 $\quad \sigma_{①} = \frac{E}{L}(q_2 - 0) = \frac{f_0 L}{4A} = 6750\,\text{psi}$

$\sigma_{①}^{nodal} = \sigma_{①}^{el}$ (no line load)



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el 2 $\quad \sigma_{②} = \frac{E}{L}(q_3 - q_2) = \frac{E}{L}\left(\frac{3}{4} - \frac{1}{4}\right)\frac{f_0 L^2}{AE} = 13{,}500\,\text{psi}$

nodal $\quad \sigma_2^{nod} = \sigma_{②} + \frac{f_2^{(2)}}{A} = 13{,}500 + 13{,}500 = 27{,}000\,\text{psi}$

$\sigma_3^{nod} = \sigma_{②} - \frac{f_3^{(2)}}{A} = 13{,}500 - 13{,}500 = 0.$

2

Accuracy: displ should be exact at nodes and nodal stresses should be exact at nodes.
Here, because $\hat{f}$ is either zero or constant (and consequently σ_{exact} is either const or linear) the nodal stresses are exact over the whole bar.

1